MATH 303 – Measures and Integration
Homework 6*

*This is the fifth homework assignment of the semester. I have given it the number 6 to match with the lecture and exercise numbering, since there was no homework to accompany lecture 5.

Please upload a pdf of your solutions by 23:59 on Monday, November 4. The assignment will be graded out of 16 points (8 for each problem). One problem will be checked for completeness and the other will be graded on correctness and quality. More details on grading, as well as guidelines for mathematical writing, can be found on Moodle.

Problem 1. Let μ be a locally finite Borel measure on $\mathbb{R}$ with distribution function F. For $x \in \mathbb{R}$, define $F(x^-) = \lim_{y \to x^-} F(y)$. The limit exists since F is an increasing function.

(a) Show that $\mu(\{x\}) = F(x) - F(x^-)$.

(b) Deduce that μ is a continuous measure if and only if F is a continuous function.

Solution: (a) Let $x \in \mathbb{R}$. For each $n \in \mathbb{N}$, the set $E_n = \left(x - \frac{1}{n}, x\right]$ has measure

$$\mu(E_n) = F(x) - F\left(x - \frac{1}{n}\right) < \infty.$$

The sets $(E_n)_{n \in \mathbb{N}}$ form a nested sequence $E_1 \supseteq E_2 \supseteq \dots$ with $\bigcap_{n \in \mathbb{N}} E_n = \{x\}$. Therefore, by continuity from above of the measure μ,

$$\mu(\{x\}) = \lim_{n \to \infty} \mu(E_n) = F(x) - \lim_{n \to \infty} F\left(x - \frac{1}{n}\right) = F(x) - F(x^-).$$

(b) Recall that the measure μ is continuous if and only if $\mu(\{x\}) = 0$ for every $x \in \mathbb{R}$. The distribution function F is always right-continuous, so F is continuous at a point $x \in \mathbb{R}$ if and only if $F(x^-) = F(x)$. Therefore, by (a), F is continuous at x if and only if $\mu(\{x\}) = 0$. It immediately follows that F is continuous on $\mathbb{R}$ if and only if μ is a continuous measure.

Problem 2. A family of sets $\mathcal{C} \subseteq \mathscr{P}(X)$ is called a *monotone class* if it is closed under countable monotone unions and intersections. That is,

- if $(A_n)_{n \in \mathbb{N}}$ is a sequence in $\mathcal{C}$ and $A_1 \subseteq A_2 \subseteq \dots$, then $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{C}$, and
- if $(B_n)_{n \in \mathbb{N}}$ is a sequence in $\mathcal{C}$ and $B_1 \supseteq B_2 \supseteq \dots$, then $\bigcap_{n \in \mathbb{N}} B_n \in \mathcal{C}$.

The goal of this problem is to prove the monotone class theorem.

(a) Show that a family of sets is a σ-algebra if and only if it is both an algebra and a monotone class.

(b) Show that the intersection of a family of monotone classes is a monotone class.

(c) Let $\mathcal{C}$ be a monotone class. Show that $\mathcal{C}' = \{E \subseteq X : X \setminus E \in \mathcal{C}\}$ is also a monotone class.

(d) Let $\mathcal{C}$ be a monotone class. Show that $\mathcal{C}_E = \{F \subseteq X : E \cup F \in \mathcal{C}\}$ is also a monotone class for every $E \subseteq X$.

Let $\mathcal{A} \subseteq \mathscr{P}(X)$ be an algebra on X, and let $\mathcal{C}(\mathcal{A})$ be the monotone class generated by $\mathcal{A}$. (This monotone class is well-defined by part (b).)

(e) Use part (c) to show that $\mathcal{C}(\mathcal{A})$ is closed under complementation.

(f) Use part (d) to show that $\mathcal{C}(\mathcal{A})$ is closed under finite unions.

(g) Deduce the monotone class theorem: $\mathcal{C}(\mathcal{A}) = \sigma(\mathcal{A})$.

Solution: (a) Suppose $\mathcal{B}$ is a σ-algebra. Then $\mathcal{B}$ is an algebra, and $\mathcal{B}$ is closed under countable unions and intersections (not only monotone ones), so $\mathcal{B}$ is a monotone class.

Conversely, suppose $\mathcal{B}$ is both an algebra and a monotone class. Since $\mathcal{B}$ is an algebra, we have $X \in \mathcal{B}$, and $\mathcal{B}$ is closed under complements. We need to show that $\mathcal{B}$ is closed under countable (not necessarily monotone) unions. Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{B}$. For $n \in \mathbb{N}$, let $F_n = \bigcup_{j=1}^n E_j$. Then $F_1 \subseteq F_2 \subseteq \dots$, and $F_n \in \mathcal{B}$, since $\mathcal{B}$ is an algebra. Thus, since $\mathcal{B}$ is a monotone class, $\bigcup_{n\in\mathbb{N}} E_n = \bigcup_{n\in\mathbb{N}} F_n \in \mathcal{B}$.

(b) Let $(\mathcal{C}_i)_{i\in I}$ be a family of monotone classes on X, and let $\mathcal{C} = \bigcap_{i\in I} \mathcal{C}_i$. Suppose $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence in $\mathcal{C}$. Then for each $i \in I$, $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence in $\mathcal{C}_i$, so $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{C}_i$, since $\mathcal{C}_i$ is a monotone class. Hence, $\bigcup_{n\in\mathbb{N}} A_n \in \bigcap_{i\in I} \mathcal{C}_i = \mathcal{C}$. A similar argument applies for decreasing sequences.

(c) We will prove that $\mathcal{C}'$ is closed under increasing unions. Closure under decreasing intersections follows similarly. Suppose $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence in $\mathcal{C}'$. Then $B_n = X \setminus A_n \in \mathcal{C}$ defines a decreasing sequence in $\mathcal{C}$, so $\bigcap_{n\in\mathbb{N}} B_n \in \mathcal{C}$. Therefore,

$$\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} (X \setminus B_n) = X \setminus \bigcap_{n\in\mathbb{N}} B_n \in \mathcal{C}'.$$

(d) Let $(A_n)_{n\in\mathbb{N}}$ be an increasing sequence in $\mathcal{C}_E$. Then $(E \cup A_n)_{n\in\mathbb{N}}$ is an increasing sequence in $\mathcal{C}$, so

$$E \cup \left(\bigcup_{n\in\mathbb{N}} A_n \right) = \bigcup_{n\in\mathbb{N}} (E \cup A_n) \in \mathcal{C}.$$

Hence, $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{C}_E$. Similarly, given a decreasing sequence $(B_n)_{n\in\mathbb{N}}$ in $\mathcal{C}_E$, the sequence $(E \cap B_n)_{n\in\mathbb{N}}$ is a decreasing sequence of elements of $\mathcal{C}$, so

$$E \cup \left(\bigcap_{n\in\mathbb{N}} B_n \right) = \bigcap_{n\in\mathbb{N}} (E \cup B_n) \in \mathcal{C},$$

whence $\left(\bigcap_{n\in\mathbb{N}} B_n\right) \in \mathcal{C}_E$.

(e) By (c), the family $\mathcal{C}(\mathcal{A})' = \{E \subseteq X : X \setminus E \in \mathcal{C}(\mathcal{A})\}$ is a monotone class. Moreover, if $A \in \mathcal{A}$, then $X \setminus A \in \mathcal{A} \subseteq \mathcal{C}(\mathcal{A})$, so $A \in \mathcal{C}(\mathcal{A})'$. Thus, $\mathcal{C}(\mathcal{A})'$ is a monotone class containing the algebra $\mathcal{A}$. By construction, $\mathcal{C}(\mathcal{A})$ is the smallest monotone class containing $\mathcal{A}$, so $\mathcal{C}(\mathcal{A}) \subseteq \mathcal{C}(\mathcal{A})'$. That is, for every $E \in \mathcal{C}(\mathcal{A})$, we have $X \setminus E \in \mathcal{C}(\mathcal{A})$, so $\mathcal{C}(\mathcal{A})$ is closed under complementation.

(f) First let $E \in \mathcal{A}$ and consider $\mathcal{C}(\mathcal{A})_E = \{F \subseteq X : E \cup F \in \mathcal{C}(\mathcal{A})\}$. Since $\mathcal{A} \subseteq \mathcal{C}(\mathcal{A})$ and algebras are closed under finite unions, we have $\mathcal{A} \subseteq \mathcal{C}(\mathcal{A})_E$. But by part (d), $\mathcal{C}(\mathcal{A})_E$ is a monotone class, so $\mathcal{C}(\mathcal{A}) \subseteq \mathcal{C}(\mathcal{A})_E$. Thus, for any $E \in \mathcal{A}$ and $F \in \mathcal{C}(\mathcal{A})$, we have $E \cup F \in \mathcal{C}(\mathcal{A})$.

Now let $F \in \mathcal{C}(\mathcal{A})$ be arbitrary. By the previous paragraph, $\mathcal{C}(\mathcal{A})_F = \{E \subseteq X : E \cup F \in \mathcal{C}(\mathcal{A})\}$ is a monotone class containing the algebra $\mathcal{A}$, so $\mathcal{C}(\mathcal{A}) \subseteq \mathcal{C}(\mathcal{A})_F$. That is, for any two sets $E, F \in \mathcal{C}(\mathcal{A})$, we have $E \cup F \in \mathcal{C}(\mathcal{A})$.

(g) By (a), every σ-algebra is a monotone class. It therefore suffices to check that $\mathcal{C}(\mathcal{A})$ is a σ-algebra. Using (a) again, we only need to check that $\mathcal{C}(\mathcal{A})$ is an algebra. Since $X \in \mathcal{A}$, we have $X \in \mathcal{C}(\mathcal{A})$. Finally, by parts (e) and (f), $\mathcal{C}(\mathcal{A})$ is closed under complements and finite unions, so $\mathcal{C}(\mathcal{A})$ is an algebra.